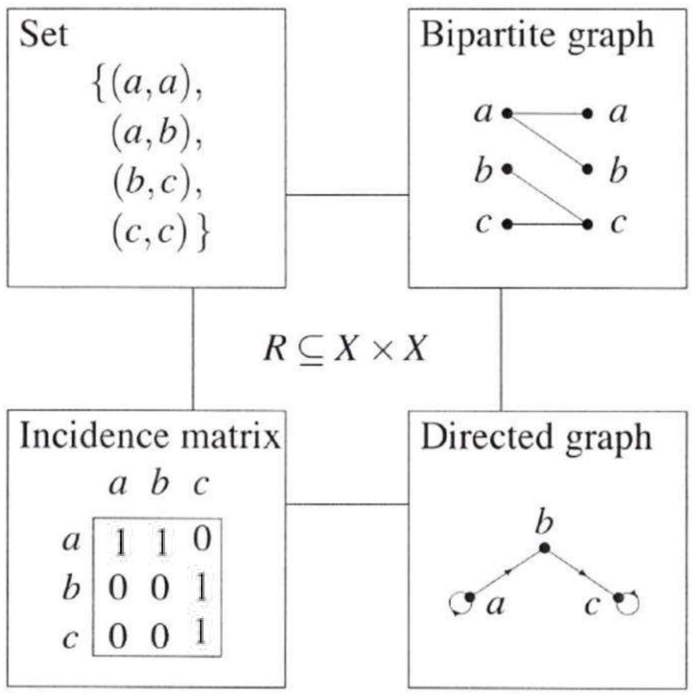
1. Binary Relations

There are four ways to look at a binary relation $R \subseteq X \times X$. A set R, as a bipartite graph G, as a directed graph D, as an incidence matrix M. For the examples we assume that X contains finite amount of elements.



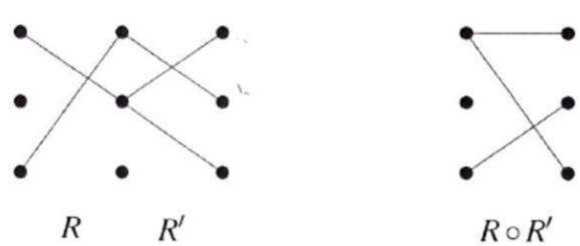
The empty set $\emptyset$ is a relation and like many other sets, they are operated upon by complement R^c, intersection $\cap$, and union $\cup$. Then there is the identity relation D_X, the inverse relation R^{-1} and the composition operation $\circ$:

$D_X = \{(x, x);\ x \in X\};$

$R^{-1} = \{(y, x);\ (x, y) \in R\};$

$R \circ R' = \{(x, z);\ (x, y) \in R \text{ and } (y, z) \in R' \text{ for some } y \in X\}$

Here is a figure that shows the composition in terms of bipartite graphs.



2. What is a poset?

A poset is short for *partially ordered set* which is a set whose elements are ordered but not all pairs of elements are required to comparable in the order. The definition has two versions of it: strict ($<$) and non-strict ($\leq$) partial order. The second version is used most of the time and we will use it here. Though it is important to note what distinguishes both of them.

Strict Partial Order	Non-Strict Partial Order
(R-) for no $x \in X$ does $(x, x) \in S$ hold (A-) if $(x, y) \in S$, then $(y, x) \notin S$ (T) if $(x, y) \in S$ and $(y, z) \in S$, then $(x, y) \in S$	(R+) for all $x \in X$ we have $(x, x) \in R$ (A) if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$ (T) if $(x, y) \in R$ and $(y, z) \in S$, then $(x, y) \in S$

(R-),(R+) = irreflexivity resp. reflexivity
(A-),(A) = antisymmetry resp. symmetry
(T) = transitivity

It is apparent that condition (R-) and (A) imply (A). This means we can replace (A-) with (A) in the definition of strict partial order. Furthermore if we now take these conditions and in terms of identity D_X, inverse R^{-1} and composition operation $\circ$ then we can see:

(R-) $D_X \cap R = \emptyset$

(R+) $D_X \subseteq R$

(A) $R \cap R^{-1} \subseteq D_X$

(T) $R \circ R \subseteq R$

Proposition 2.1 Let X be a set

a) If S is a strict partial order on X, then $S \cup D_X$is a non-strict partial order on X
b) If R is a non-strict partial order on X, then $R \backslash D_X$is a strict partial order on X
c) These two are mutually inverse.

3. Preorders

When we weaken the definition of a partial order and take out one of its condition, then we get Preorders. Preorders are only reflexive and transitive. They are also called *partial preorder* or *pseudo-order*. In preorders its permitted that distinct elements x and y satisfy $(x, y) \in R$ and $(y, x) \in R$.

Proposition 3.1

Let R be a partial preorder on X. Define a relation $\sim$ on X by the rule that $x \sim y$ if and only if (x, y), $(y, x) \in R$. Then $\sim$ is a equivalence relation on X. Moreover, if $x \sim x'$ and $y \sim y'$, then $(x, y) \in R$ if and only if $(x', y') \in R$. Thus, R induces in a natural way a relation $\bar{R}$ on the set $\bar{X}$ of equivalence classes of X; and $\bar{R}$ is a non-strict partial order on $\bar{X}$.

4. Properties of posets

An element of a poset (X, R) is called *maximal* if there is no element $y \in X$ satisfying $x <_R y$. At the same time, x is *minimal* if no elements satisfy $y <_R x$. In general, there may be no maximal element or there may be more than one in a poset. However in a finite poset, there can at last be one maximal element which can be found as follows: we simply chose one element that is not the maximal and then replace it by an element $y$which satisfies $x <_R y$. We repeat this until we find the maximal element. It is important to end the process due to the irreflexivity and transitivity rule which tells that any element cannot be revisited. Simultaneously, a finite poset must contain minimal elements too.

A *chain* in a poset (X, R) is a subset C of X which is totally ordered by the restriction of R that is, a totally ordered subset of X. An *antichain* is a set A of pairwise incomparable elements.

There are infinite posets (like $\mathbb{Z}$) and they do not have to contain maximal elements. *Zorn's Lemma* gives a sufficient condition for maximal elements to exist:
Let (X, R) be a poset in which every chain has an upper bound. Then X contains a maximal element.

It's important to keep the two terms in mind as the height $h(X)$ of the poset is determined by the largest cardinality of a chain and its width $w(X)$ from the largest cardinality of an antichain.

Theorem 4.1

Let (X, R) be a finite poset. Then let X break into $w(X)$ chains.

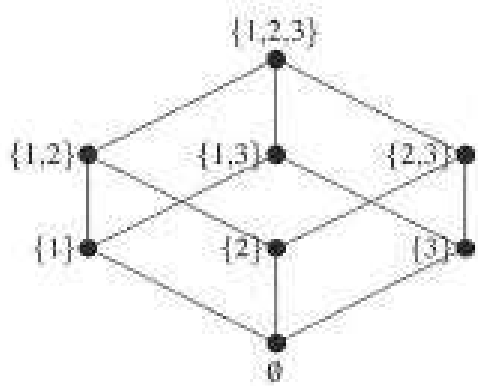
Up-set in a poset (X, R) is a subset of Y of X such that, if $y \in Y$ and $y <_R z$, then $z \in Y$. The set of minimal elements in an up-set is an antichain.

Down-sets are defined dually. The compliment of an up-set is a down –set and there are equal amounts of them.

5. Hasse diagrams.

Proposition 5.1

Let (X, R) be a locally finite poset and $x, y \in X$. Then $x <_R y$ iff there exists elements $z_0, \ldots, z_n$ (for some non-negative integer n) such that $z_0 = x$, $z_n = y$ *and* z_{i+1} covers z_i for $i = 0, \ldots, n-1$.



6. Lattices

A lattice is a poset (X, R) with two properties:

- X has an upper bound 1 and a lower bound 0;
- for any two elements $x, y \in X$, there is a least upper bound and a greatest lower bound of a set $\{x, y\}$.

In a lattice, we denote the least upper bound of $\{x, y\}$ by $x \vee y$ and the greatest lower bound by $x \wedge y$. The lattice is commonly regarded as a set with two distinguished elements and two binary operations rather than a special kind of poset.
Lattices are expressed in axioms in terms of two constants 0 and 1 and the two operations ∧ and ∨. The following axioms in the next page are not all independent. In finite lattices we don't need to specify 0 and 1 separately since 0 is just the meet of all elements in the lattice and 1 is their join.

Proposition 6.1

Let X be a set, ∧ and ∨ two binary operations defined on X, and 0 and 1 two elements of X. Then $(X,\wedge,\vee\ ,0,1)$ is a lattice iff the following axioms are satisfied:

- Associativity: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$;
- Commutativity: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;
- Idempotent laws: $x \wedge x = x \vee x = x$;
- $x \wedge (x \vee y) = x = x \vee (x \wedge y)$;
- $x \wedge 0 = 0, x \vee 1 = 1$.

A sublattice of a lattice is a subset of elements containing 0 and 1 and closed under the operations ∧ and ∨. It's a lattice in its own right. The examples shown below are of lattices.

Examples:

- The subset of a (fixed) set:
  $A \wedge B = A \cap B$
  $A \vee B = A \cup B$

- The subspaces of a vector space:
  $U \wedge V = U \cup V$
  $U \vee V = span(U \cup V)$

- The partisal pseudo-orders on a set:
  $R \wedge T = R \cap T$
  $R \vee T = \overline{R \cup T}$

7. Distributive and modular lattices

A lattice is distributive if it satisfies the distributive laws (D) and it is modular is it satisfies the modular law (M).

(D) $x \wedge (y \vee z) = (x \wedge y)(x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y)(x \vee z)$for all x, y, z**.**
(M) $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all x, y, z such that $x \leq z$.

Theorem 7.1

A lattice is modular iff it does not contain the lattice N_5 as a sublattice. A lattice is distributive iff it contains neither the lattice N_5 nor the lattice N_3 as a sublattice. In the same way as in Proposition 6.1, we are able to describe distributive axioms axiomatically.

Figure 7: Two lattices

Proposition 7.2

Let X be a set, ∧ and ∨ two binary operations defined in X, and 0 and 1 two elements of X. Then $(X,\wedge,\vee,0,1)$ is distributive lattice iff the condition (D) and the following axioms are satisfied:

- Idempotent law: $x \wedge x = x$;
- $x \wedge 1 = 1 \vee x = 1$;
- $x \wedge 0 = 0 \vee x = 0$

Theorem 7.3

(a) Let (X, R) be a finite poset. Then the set of down-sets in X, with the operations ∩ and ∪ and distinguished elements $0 = \emptyset$ and $1 = X$, is a distributive lattice.

(b) Let L be a finite distributive lattice. Then the set X of non-zero join-irreducible elemets of L is a sub-poset of L. A non-zero element $x \in L$ is called join-irreducible if, whenever $x = y \vee z$, we have $x = y$ or $x = z$.

(c) These two operations are mutually inverse. ■